

Last time

$$G \curvearrowright S$$

$$\begin{matrix} \psi \\ s \end{matrix} \rightsquigarrow O_s \text{ the orbit}$$

$H \leq G$, the stab. of s

$$\begin{matrix} \text{Then } G/H & \xrightarrow{\pi} & O_s & \text{bijection} \\ \uparrow & & \uparrow \\ G & & G \end{matrix}$$

compatible w/ gp actions

Propⁿ S a finite set, $G \curvearrowright S$, G_s the stab of $s \in S$, O_s the orbit. Then

$$|G| = |G_s| \cdot |O_s|$$

$$\text{Pf: } |O_s| = \underbrace{|G/G_s|}_{\text{the set of cosets}} = [G : G_s] = \frac{|G|}{|G_s|} \quad \square$$

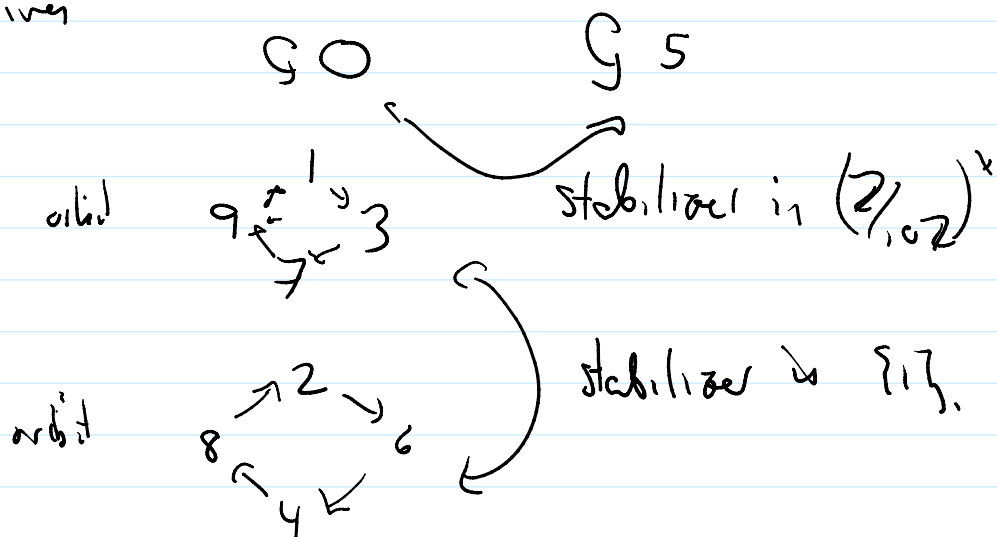
• In other words $|O_s| = [G : G_s]$.

• Also: $|S| = |O_1| + |O_2| + \dots + |O_k|$ is the sum of the sizes of the various orbits

Later: Will use this to prove: every gp whose order is p^r has a non-trivial center

Example $(\mathbb{Z}_{102})^{\times} \triangleleft \mathbb{Z}_{102}$

has representatives
1, 3, 7, 9

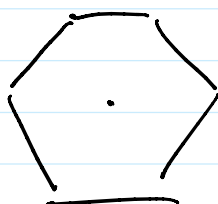


$$|S| = 10 = (1 + 4 + 4 + 1)$$

Example

$$G = D_6$$

$G \triangleleft$

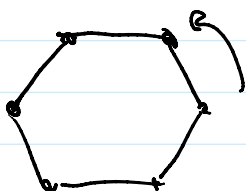


regular hexagon

$$G = \{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$$



y is reflection across this axis

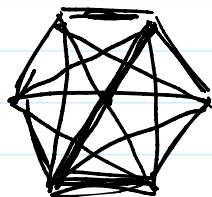


x is rotation by $\frac{2\pi}{6}$

S = set of line segments connecting two vertices in the hexagon

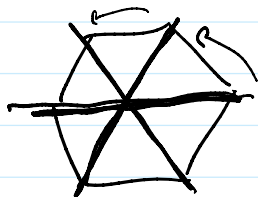
$$|S| = 15$$

6 edges
3 diagonals
6 semi-diagonals

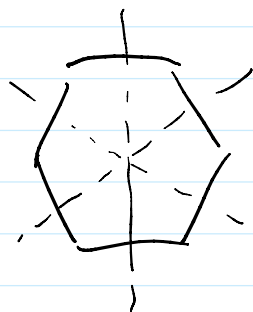


Understand the orbits and stabilizers for the G action on S

Other elts of G



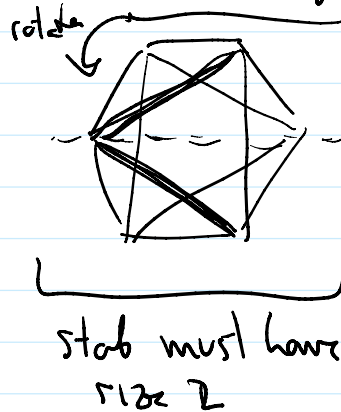
G contains 3 reflections thru diagonals



3 reflections thru those lines

$$|S| = 15 = \underbrace{(3 \text{ diags})}_{\text{orbit}} + \underbrace{(6 \text{ semi diags})}_{\text{orbit}} + \underbrace{(6 \text{ edges})}_{\text{orbit}}$$

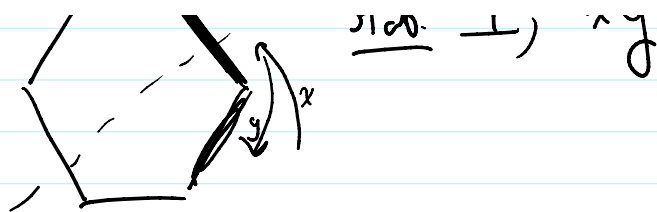
stab. of a
diagonal
must have
size 4



stab must
have size 2



Stab: 1, x_y



stab: $1, y, x^3, x^3y$

Semi direct products

- Have gps $H, K \rightsquigarrow$ direct product $H \times K$
- generalization of this construct; involves gp actions
- Setup: If $K \leq H$ acts by automorphisms, meaning

$$k \bullet (h_1 h_2) = (k \bullet h_1) \cdot (k \bullet h_2)$$

$\rightsquigarrow$ can construct a new gp within an $H \rtimes K$

Recall: $M_n \leq \mathbb{R}^n$, gp of isometries

$T_n \leq M_n$, the subgroup of translations

$$T_n = \{t_v : v \in \mathbb{R}^n\} \cong (\mathbb{R}^n, +).$$

$$T_n = \{t_v : v \in \mathbb{R}^n\} \approx (\mathbb{R}^n, +).$$

Facts

- $M_n = T_n \rtimes O_n$, O_n the orthogonal gp
- $\varphi: M_n \rightarrow O_n$ is a sp homom, with kernel T_n
 $t_v p \mapsto p$

$$\text{Multiplication: } (t_v p)(t_{v'} p') = t_v t_{p(v')} p p'$$

one is using an action of O_n on $T_n \approx \mathbb{R}^n$

- This is an example of a semidirect product
- This is a special case of the general setup

The construction

- Suppose I have sps H, K
- $\phi: K \rightarrow \text{Aut}(H) = \{\text{set of isoms } H \rightarrow H\}$.

In other words, $K \curvearrowright H$ by gp isoms.

Will write: ϕ_k to be the value of ϕ applied to $k \in K$

$$\Rightarrow \phi_k \in \text{Aut}(H)$$

$$\Rightarrow \phi_k(h_1 h_2) = \phi_k(h_1) \phi_k(h_2).$$

Define $H \rtimes_\phi K$ to be the set of pairs (h, k)
 with $h \in H, k \in K$

with multiplication

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 \phi_{k_1}(h_2), k_1 k_2)$$

Then a) $H \rtimes_{\phi} K$ is a gp

b) The map $H \rtimes_{\phi} K \rightarrow K$ given by $(h, k) \mapsto k$ is a surjective gp homom with kernel H

c) If $\phi: K \rightarrow \text{Aut}(H)$ is the identity map, then

$H \rtimes_{\phi} K \cong H \times K$, the direct product of $H \times K$